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ABSTRACT

Forty-three first-grade children, who had received no formal instruction in addition and subtraction, were individually administered 20 problems that could be solved using addition or subtraction. The problems were selected to represent the following semantic types: joining, separating, part-part-whole, comparison, and equalizing. Responses were coded in terms of appropriateness of strategy, correct or incorrect answer, type of error, mode of representation, and solution strategy. For every problem but the two addition comparison problems, over 70% of the subjects chose a correct strategy. There were very few systematic errors. Only 15 of the 860 responses involved the wrong choice of operation. The majority of solutions involved the use of concrete objects, but a significant number used fingers or did not use any physical representation. Contrary to previous analysis of children's solution processes for addition and subtraction problems, these results suggest that children do not transform problems so they can apply a single strategy. Rather, they have a rich repertoire of strategies which they apply directly to a problem based on its semantic structure. The results also suggest that verbal problems may be an appropriate context to introduce addition and subtraction operations. (Author/MK)

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by
Thomas P. Carpenter,
James Hiebert, and
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and Development Center
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by

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Technical Report No. 1 of
the Studies in Mathematics Series

Report from the Project on
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Abstract

Forty-three first grade children who had received no formal instruction in addition and subtraction were individually administered 20 problems that could be solved using addition or subtraction. The problems were selected to represent the following semantic types: joining, separating, part-part-whole, comparison, and equalizing. Ten problems were presented physically using sets of concrete objects, and 10 corresponding problems were presented through verbal problem situations. Number triples for all problems were selected so that the sum of the two addends was between 10 and 17. For all problems physical objects were available to aid in the solution.

Responses were coded in terms of appropriateness of strategy, correct or incorrect answer, type of error, mode of representation, and solution strategy. For every problem but the two addition comparison problems, over 70% of the subjects chose a correct strategy. There were very few systematic errors. Only 15 of the 860 responses involved the wrong choice of operation. The majority of solutions involved the use of concrete objects, but a significant number used fingers or did not use any physical representation.

For the verbal problems, children's solution processes modeled the action or relationships described in the problems. Thus, they

used a variety of different strategies depending on the semantic structure of the problem. These strategies are consistent with a proposed model of problem structure.

For the concrete problems, strategies were principally determined by the characteristics of the sets of cubes in problems. They generally operated on the set of cubes available from the problem statement rather than attempting to model the action.

Contrary to previous analysis of children's solution processes for addition and subtraction problems, these results suggest that children do not transform problems so they can apply a single strategy. Rather they have a rich repertoire of strategies which they apply directly to a problem based on its semantic structure. These results also suggest that verbal problems may be an appropriate context to introduce addition and subtraction operations.

Introduction

A major goal of mathematics instruction is to teach children to apply their mathematical skills to solve problems. It is frequently assumed that children must first master computational skills before they can begin to apply them to the solution of problems. However, although it is reasonable to assume that children will not be able to apply formal algorithms without instruction, it has been clearly demonstrated that children develop a variety of informal strategies for solving mathematical problems independent of instruction (c.f., Ginsburg, 1977; Resnick, Note 1). In fact, many of the informal strategies are more sophisticated and demonstrate more insight than the formal procedures that are a part of instruction. This raises the hypothesis that, rather than depending on a prior knowledge of computational skills, simple problems may give meaning to basic mathematics operations. To a limited degree, most initial instruction in the four basic operations on whole numbers is based on this hypothesis. Almost all major mathematics programs initially introduce addition, subtraction, multiplication, and division through some sort of physical or pictorial representations. However, the range of problem types used as examples in most instructional programs is very narrow. For example, subtraction is almost always initially represented in terms of a separating model in which a subset of a given set is removed;

addition and subtraction are almost exclusively introduced with physical models or pictures that directly represent joining or separating rather than with problems that require children to construct representations of the operation themselves. These instructional decisions are based on very limited evidence regarding the appropriateness of different types of problem situations as initial models for the basic operations.

This study focused on children's initial concepts of addition and subtraction as shown by their ability to solve selected problems representing addition and subtraction operations. The working hypothesis of the study was that prior to formal instruction many children can solve a variety of different problems involving addition and subtraction operations. Furthermore, they develop different strategies for solving different problems. By identifying the processes children use to solve different problems, the study attempted to gain a clearer picture of children's initial concepts of addition and subtraction as well as to provide some insights into their problem-solving abilities.

This study was not carried out in isolation. Rather, it is part of a series of short- and long-term investigations being carried out by the Mathematics Work Group of the Wisconsin Research and Development Center for Individualized Schooling. This set of studies has three major objectives: (a) to describe the development of addition and subtraction concepts and skills in children and to identify how this development is related to the development of underlying cognitive

skills, (b) to identify changes in performance on these concepts and skills that result from specific instruction, and (c) to ascertain the effects of certain teacher actions on pupil engagement and performance on addition and subtraction concepts and skills.

The present study fits under the first major objective. In describing the development of addition and subtraction it is necessary to characterize processes and strategies children use in solving selected addition and subtraction problems as well as to identify the errors that result from applying inappropriate or incorrect strategies. This characterization and identification was one of the major aims of this study.

Background

Addition and subtraction problems presented to young children can be grouped into two large categories. One category is the purely mathematical presentation of symbolic sentences in either horizontal or vertical form. The other category is nonsymbolic problems in which the numbers are measures of entities described in the problem situation. Included in this second category are the so-called "story problems" presented in most textbooks. For the study presented in this report, only the latter type of problem was included.

Two dimensions can be identified that divide nonsymbolic addition or subtraction problems into four distinct classes. The first dimension is based upon whether an active or static relationship between sets or objects is implied in the problem. Some problems may contain an explicit reference to a completed or contemplated action causing a change in the size or position of problem entities. For example, "Sue had 8 apples in a basket. Then she put 6 more apples in that basket. How many apples did she have altogether?" Contrasted to such situations are those in which no action is implied; that is, there is a static relationship. As an example, consider, "There are 7 apples in a basket. Four are red and the rest are green. How many of the apples are green?"

The second dimension involves a set inclusion or set-subset relationship. In certain problems two of the entities involved in the problem are necessarily a subset of the third. In other words, either the unknown quantity is made up of the two given quantities, or one of the given quantities is made up of the other given quantity and the unknown. For example, consider the following problem: There are seven children on the playground. Three are boys and the rest are girls. How many are girls? The set of boys and the set of girls are subsets of the set of children. The alternative is that one of the quantities is disjointed from the other two. For another example, consider the following problem: There are seven girls and three boys on the playground. How many more girls than boys are there? In this problem, removing a set of three girls and counting the number of girls in the remaining set of four girls gives the answer. The distinction between this problem and the preceding one is that the set of boys is disjointed from all of the sets of girls involved.

The relationship between the action/static dimension and the set-subset dimension can be represented in a two-by-two matrix (Figure 1). A characterization and examples of the problems corresponding to the cells of this matrix follows.

Joining and Separating

These two problem situations arise frequently in early mathematics instruction because they are generally easy to understand, and they tend to be familiar situations for young children. Joining is the

	active	static
set inclusion	joining and separating	part-part-whole
no set inclusion	equalizing	comparison

Figure 1. Types of nonsymbolic problem situations.

process of putting together two entities to form a single entity. The action is incremental because the first set is made larger by the adjoining of the second set. For example:

1. Wally has 3 pennies. His father gives him 6 more pennies. How many pennies does he have altogether?

By varying the unknown quantity it is also possible to generate joining problems that represent the arithmetic operation of subtraction.

2. Wally had 3 pennies. His father gave him some more pennies. Now he has 9 pennies. Haw many pennies did his father give him?

Separating is the process of breaking up or separating a single entity into two subentities and then removing one of those two. For separating, the action results in a decrease in size of the original set. An example is:

3. Fred had 11 candies. He gave 4 of them to Kathy. How many candies does he have left?

Part-part-whole

This is a static relationship that exists between an entity and its two component parts. Some examples of part-part-whole follow.

4. There are 3 boys and 8 girls in the dancing class. How many children are there altogether?

5. Maria has 9 toy cars. Four are red and the rest are blue. How many blue cars does she have?

Comparison

Comparison problems involve the static relationship of order existing between two disjoint entities.

6. Mark has 5 balloons. His sister Connie has 12 balloons. How many more balloons does Connie have than Mark?
7. Joe has 13 records. Mike has 6 more records than Joe. How many records does Mike have?

Equalizing

Equalizing problems share characteristics of both joining/separating and comparison problems. There is implied action on a given entity but a comparison is also involved. Equalizing is the process of changing one of two entities, so that the two are then equal on some particular attribute. For example:

8. There are 4 boys and 7 girls on the class basketball team. How many more boys have to be added to the team so there will be the same number of boys and girls?

Other attempts to characterize the different classes of addition and subtraction problems are generally consistent with the above analysis. Greeno and associates (Note 2) identify three distinct schemata believed necessary and sufficient for understanding all

problems that are solved by a single operation of addition or subtraction. The first is called Cause/Change and encompasses situations in which some event changes the value of a quantity. This essentially corresponds to the joining and separating category described above. For Greeno, the joining situation is one in which the direction of the change causes an increase in some given quantity and separating is the situation for which the direction of change causes a decrease in a given quantity. The second schema proposed by Greeno (Note 2) is Combination, which corresponds directly to the part-part-whole problem type described earlier. The final schema, Comparison, is related to the description of the same name given earlier.

Nesher and Katriel (Note 3) have also carried out an analysis of verbal problem types. They have identified the same three basic categories as Greeno (Note 2). However, they use the term Dynamic Description for the Cause/Change or joining/separating category, Static Description for the Combination or part-part-whole category, and the familiar term Comparison for the third identified problem type.

Any classification system is somewhat arbitrary, and problems can be classified in different ways by focusing on different dimensions. In a pair of studies carried out at the Wisconsin Research and Development Center for Cognitive Learning, Steffe (1970), who studied performance on addition problems, and LeBlanc (Note 4), who investigated subtraction problems, simply differentiated between action and no action in verbally stated problems. They used the term "transformation"

to characterize action being carried out on the sets described in the problem situations.

Equalizing problems were first identified as a unique class of problems by the Developing Mathematical Processes (DMP) program developed at the Wisconsin Research and Development Center (Romberg et al., 1974). In DMP, equalizing is used as the vehicle for introducing the operational symbols for addition and subtraction and as the initial problem situations for which children are asked to write number sentences.

A more thorough analysis of problem types has been reported elsewhere (Moser, Note 5). He has identified a third dimension along which problems can be categorized. This is an order dimension of either making or being larger or smaller. Thus, the joining situation, for example, would fall into the making-larger category and separating would come under the making-smaller category. Equalizing can be accomplished by making the smaller of two compared sets larger or by making the larger of two compared sets smaller. In a similar fashion, the difference between two compared sets can be characterized by either how much larger the bigger set is or by how much smaller is the lesser of the two sets. This larger/smaller dimension does not appear to apply, however, to the static part-part-whole situations.

A number of studies have investigated children's solutions to verbal problems. Because the presence of manipulative aids was a part of the present study, several studies that relate to these aids should be briefly mentioned. Both Steffe (1970) and LeBlanc (Note 4) found

that the presence of aids, both physical and pictorial, contributed to significantly better performance in the solution of addition and subtraction problems. In a recent study carried out with kindergarten children, Ibarra and Lindvall (Note 6) found that the degree of concreteness accompanying the presentation of verbal problems significantly affected the proportion of students responding correctly to those problems.

There is a scarcity of research or analysis dealing with the processes children use to solve simple verbal addition or subtraction problems. Greeno (Note 2) has hypothesized that certain types of problems are associated directly with addition or subtraction operations. Other types of problems are transformed to one of the representations that is directly associated with an operation before a solution is attempted. In general, the canonical forms (problems that are naturally represented as $a + b = \square$ or $a - b = \square$) are directly translated to addition or subtraction operations while noncanonical forms (e.g., missing addend problems) are first transformed to part-part-whole representations. Greeno has little empirical support for this analysis. Most research on these types of verbal problems also has focused on level of difficulty of different problems rather than solution processes (Nesher & Katriel, Note 4; Steffe, 1970). This research has found that static problems are generally more difficult than corresponding problems involving action. If all noncanonical problems must first be represented as part-part-whole problems, one

would hypothesize that part-part-whole problems would have fewer errors since they would require one fewer transformation. The fact that they are more difficult than corresponding problems involving action casts some doubt on the validity of Greeno's analysis. In any case a more direct measure of children's solution processes seems to be needed before any conclusions can be drawn.

A number of studies have investigated the strategies that children use to solve open addition and subtraction sentences. In a study involving third-graders, Grouws (1974) individually interviewed subjects and coded responses and strategies used to solve four different types of open sentences. Algorithmic behavior, recall of basic facts, and counting were the most frequent solution methods.

The largest collection of such studies have relied upon response latencies to infer what strategy a child applies to a given type of problem. The general technique involves breaking the operations down into a series of discrete steps, in this case counting by ones. It is assumed that the time required to solve a given problem using a particular strategy is a linear function of the number of steps needed to reach the solution. By finding the best fit between response latencies for subjects solving a variety of problems of a given type and the regression equations of possible solution strategies, the most appropriate model is inferred. For addition, three basic strategies have been identified (Groen & Parkman, 1972; Suppes & Groen, 1967). To calculate the answer to 3 + 5 = ?, the most basic strategy involves counting to 3 and then counting on 5 more. A somewhat more sophisti-

cated and efficient strategy is to start counting at the first number. In this case it would mean starting at 3 and counting on 5 more. The most sophisticated and efficient strategy is to start counting at the larger of the two numbers. In the above problem this would mean starting at 5 and counting on 3 more. For sums less than 10, this last strategy provides the best model of first-graders' responses (Groen & Parkman, 1972).

As part of a similar analysis of subtraction, two basic strategies were hypothesized (Woods, Resnick, & Groen, 1975). To solve 9 - 6 = ?, children might count down 6 units from 9, or they might count up from 6 until they reach 9 and keep track of the number of units counted. For this particular problem the second strategy would require fewer steps, while the counting down strategy would be more efficient for 8 - 2 = ?. The results of this study indicate that by the second grade four-fifths of the children used a choice strategy by which they choose the most efficient of the two strategies and by the fourth grade the responses of all children best fit a model predicted by such a strategy (Groen & Parkman, 1972).

These data indicate that as children mature they develop more sophisticated and efficient counting strategies. Furthermore, the results of another study indicate that these strategies are developed independent of instruction, and that the strategies that children construct for themselves are frequently more sophisticated and efficient than the ones they are taught (Groen & Resnick, 1977).

There have been some studies that focused more on children 's errors or item difficulties. Although their study focused on primary children's incorrect procedures in solving open sentences, Lindvall and Ibarra (Note 8) do report on certain errors associated with the solution of verbal addition and subtraction problems which represented noncanonical situations. When the problems were of the missing-addend type, the predominant error was to use the wrong operation and add the two given numbers. When the problems were related to an open sentence of the form $a - \square = c$ or $\square - b = c$, the greatest tendency is to report one of the given numbers as the answer.

Method

Tasks

One of the major variables included in the study was the problem structure as defined by the four major classes of problems described earlier. Problems to be included in the study were selected from each of the four basic classes of problems. By varying the unknown quantity or the nature of the action as in the joining/separating class, both addition and subtraction items can be represented in each of the four classes. For example, consider the part-part-whole problem class. If the problem were stated as, "There are some children on the playground. Six are boys and eight are girls. How many children are there altogether?", the operation required to solve the problem would be the addition of 6 and 8. On the other hand, if the problem were stated as "There are 14 children on the playground. Six are boys and the rest are girls. How many girls are on the playground?", then the operation of subtracting 6 from 14 would be required to determine the solution.

It was not feasible to include all possible forms of problems in each class in the time available for testing. Furthermore, some of the forms lend themselves less well than others to natural problem situations. Consequently, two different problems were selected from each of the part-part-whole, comparison, and equalizing classes and four

from the joining/separating class. One addition and one subtraction problem were selected to represent the part-part-whole and comparison classes. Since equalizing problems most naturally represent subtraction operations, two subtraction problems were selected from this class. One involved increasing the smaller quantity, and the other involved decreasing the greater quantity. Two distinct types of action are included in the joining/separating class, joining and separating. Since these are the most commonly used problems in elementary school mathematics programs, more than two problems were needed to adequately represent this class. The final decision was to include one joining addition problem, two joining missing-addend problems, and one separating problem. Two missing addend problems were included because two distinct forms of this type were identified, and it was not clear which was most representative.

A second major variable in this study was the mode of presentation. We decided to employ two modes, concrete and verbal. Thus, for each of the 10 types of problems, a verbal problem and a problem involving action or relationships between sets of cubes were generated. The verbal problems are presented in Table 1 and the concrete problems are listed in Table 2. The concrete problems modeled the action in the corresponding verbal problems as closely as possible. All problems were constructed to provide relatively simple examples of their type while controlling for factors such as syntax, vocabulary, sentence length, and familiarity of problem situations.

Table 1

Verbal Problems

Addition

1. Joining

   Wally had a pennies. His father gave him b more pennies. How many pennies did Wally have altogether?

2. Part-part-whole

   Some children were ice-skating. a were girls and b were boys. How many children were skating altogether?

3. Comparison

   Ralph has a pieces of gum. Jeff has b more pieces than Ralph. How many pieces of gum does Jeff have?

Subtraction

4. Separating

   Leroy had a pieces of candy. He gave b pieces to Jenny. How many pieces of candy did he have left?

5. Joining (1)

   Susan had a books. Her teacher gave her some more books. Now she has c books altogether. How many books did Susan's teacher give her?

6. Joining (2)

   Kathy had a toys. How many more does she need to have c toys altogether?

7. Part-part-whole

   There are c children on the playground. a are boys and the rest are girls. How many girls are at the playground?

8. Comparison

   Mark won a prizes at the fair. His sister Connie won c prizes. How many more prizes did Connie win than Mark?

(continued)

Table 1 (continued)

9. Equalizing (+) Joan picked a flowers. Bill picked c flowers. What could Joan do so she could have as many flowers as Bill? (Suggest, if necessary, that she pick some more.) How many more would she need to pick?
10. Equalizing (-) Fred has a marbles. Betty has c marbles. What could Betty do so she would have as many marbles as Fred? (Suggest, if necessary, giving some away.) How many would she need to get rid of?

Table 2

Concrete Problems

Addition

1. Joining

   Subject is asked to count separate sets of a red cubes and b white cubes. Cubes are physically combined and subject is asked how many cubes there are altogether.

2. Part-part-whole

   Subject is asked to count a red cubes and b white cubes in a mixed set. Subject is then asked how many cubes there are altogether.

3. Comparison

   Subject is asked to count set of a red cubes. Subject is then asked to determine how many white cubes would be in a set which had b more white cubes than red cubes.

Subtraction

4. Separating

   Subject is asked to count set of a red cubes. Subject is then asked to determine how many cubes would be left if b cubes were removed.

5. Joining (1)

   Subject is asked to count set of a white cubes. A second set of white cubes is combined with the first to make one set of c white cubes. Subject is asked to determine how many cubes were added to the first set.

6. Joining (2)

   Subject is asked to count set of a white cubes. Subject is then asked to determine how many white cubes must be added to make a set of c white cubes.

(continued)

Table 2 (continued)

7. Part-part-whole

   Subject is presented with sets of a white cubes and b red cubes and is asked to count total number of cubes and number of white cubes. Subject is then asked to determine number of red cubes.

8. Comparison

   Subject is asked to count sets of a red cubes and c white cubes. Subject is then asked to determine how many more white cubes there are than red cubes.

9. Equalizing (+)

   Subject is asked to count sets of a red cubes and c white cubes. Subject is asked to determine what must be done to the red set to make as many red cubes as white cubes. Subject is then asked to determine how many red cubes must be added to make the sets equal.

10. Equalizing (-)

    Subject is asked to count sets of a red cubes and c white cubes. Subject is asked to determine what must be done to the white set of make as many white cubes as red cubes. Subject is then asked to determine how many white cubes must be removed to make the sets equal.

The number triples for the problems were selected to conform to the following specifications: (a) each of the addends was greater than 2 and less than 10, (b) their sum was greater than 10 and less than 16, and (c) the absolute value of the difference between the two addends was greater than one. These rules generated the following set of 10 triples: (3,8,11), (3,9,12), (4,7,11), (4,8,12), (4,9,13), (5,7,12), (5,8,13), (5,9,14), (6,8,14), (6,9,15). This number range was selected because the numbers were small enough so that the problems could be conveniently modeled using concrete objects but were large enough so that is was unlikely that many children would have already learned the addition or subtraction combinations. It was also more likely that the children's strategies would be observable with numbers of this size than with smaller numbers. Doubles and near doubles were eliminated because it was hypothesized that children may operate differently with those combinations (cf. Groen & Parkman, 1972).

Although the 10 number triples are as homogeneous as possible, it is still conceivable that differences between triples may account for variability in children's performance. Therefore the number triples were equally distributed over the set of problems so that each number triple was paired with each problem either four or five times. In order for each triple to be paired with each problem exactly the same number of times, the number of subjects would have had to be a multiple of 10. Thus, each subject received each number triple exactly once within the set of verbal problems and once within the set

of concrete problems, but different subjects received different combinations for each problem. For each subject, pairings were made so that the verbal problems contained the same number combination as the corresponding concrete problem. This made the two problems as comparable as possible.

In presenting the two addends in the addition problems, we decided to present the smaller addend first. A child who realizes that counting on from one of the given numbers is more efficient than always beginning a counting sequence with "one, two, three, . . .," would probably have the tendency to count on from the first number presented. The more sophisticated child would realize further that counting would be more efficient if begun with the larger of the two given numbers. If the larger addend was presented first, it would be next to impossible to determine if the child had chosen to count on from the larger number or simply from the first number.

Another consideration in number presentation is which number should be the unknown in a subtraction problem. The data of Woods et al. (1975) indicate that although the problems 8 - 2 = ☐ and 8 - 6 = ☐ originate from the same number triple they may generate significantly different methods of solution. If this conclusion is correct, choosing the larger of the two addends as the unknown would tend to bias responses in favor of a separating or counting back strategy whereas choosing the smaller number would create a bias in favor of a counting up strategy. Since the dilemma is unavoidable, we decided that it was more important to be consistent between problems so that only the

structure of the problem varied. Consequently the larger addend was selected for the unknown in all subtraction problems.

Subjects

The subjects for the study consisted of the 43 children in the two first-grade classes of a parochial school that draws students from a predominantly middle class area of Madison, Wisconsin. Mathematics instruction in both classes consisted of topics 15 through 22 of the Developing Mathematical Processes (DMP) program (Romberg, Harvey, Moser, & Montgomery, 1974). At the time of testing in early February, only two arithmetic topics had been covered, Writing Numbers and Comparison Sentences. The topic of Comparison Sentences introduces the notion of a mathematical sentence, though at this point it only deals with representing a static relation (equality) between two numbers. The other six topics dealt with measurement and geometry. Thus, at the time the children were tested, no formal instruction in symbolic representation of addition and subtraction had been given. On the other hand, several lessons had been presented involving joining, separating, part-part-whole and comparison problems. In those instances, modeling with objects to determine the solutions had been suggested.

Procedures

This study relied upon individual interviews with children to identify the processes they were using to solve each of the problems. Ginsburg (1976) has made a strong case that this type of clinical

technique is the most appropriate for assessing children's mathematical behavior. Each problem was individually administered to each subject by one of two experimenters.

For the concrete problems the appropriate sets were constructed by the experimenter using red and white Unifix cubes. Subjects were instructed to count the elements in each set. If subjects made a counting error, they were instructed to check their result. After subjects had determined the number of elements in the sets, the action or relationship specified by the problem was described by the experimenter, and the subjects were asked to solve the problem. Extra cubes were available if subjects needed them to solve the problem.

The verbal problems were read to the subjects by the experimenter. Problems were reread as often as necessary so that ability to remember numbers or relationships in the problems was not a factor. A set of cubes identical to those used in the concrete problems was available to the subjects. They were encouraged to solve the problem without the cubes but were told to use the cubes if they needed them or were not sure of their answer. There was not strong pressure either to use the cubes or to solve the problems without them, but if subjects were floundering they were reminded that they could use cubes to find the answer.

If a solution process that a subject used was obvious, the experimenter coded the response and went on to the next problem. If how a subject had found an answer was not immediately visible, the subject

was asked to describe how the answer was found. The experimenter continued questioning until the subject's strategy was apparent or it was clear that no explanation was forthcoming. The testing required two sessions that lasted 15 to 20 minutes each. Half the subjects received the 10 concrete problems in the first session and the 10 verbal problems in the second session. For the other half this order of administration was reversed. Subjects were randomly assigned to these administration conditions. In most cases, the two sessions were separated by at least one day. For some subjects, the two sessions occurred on consecutive days. No subject received both sessions on the same day.

The order of the tasks within the concrete and verbal groups was randomized for each subject. Thus, each subject received a different sequence of problems, but each subject received the concrete problems in the same order the verbal problems were presented.

Results

During the individual interviews, the interviewers focused on four major categories of responses: (a) the mode of representation used by the child in generating a solution; (b) the strategy used to generate the solution; (c) whether the solution was correct or not; and (d) when appropriate, the type of error made. Each of these categories will be discussed in the first part of this section.

Mode of Representation

In the interview setting, sets of cubes were always present. Since paper and pencil were not available, symbolic or pictorial representations were not possible. Two basic modes of physical representation were used by the children, cubes and fingers.

(C) Cubes — Although no coding differentiation was made, there were two major ways in which cubes could be used. First, the cubes were set out to represent the actual sets in the problem situation. For example, if the problem dealt with a situation such as, "Wally has 3 pennies and his father gives him 6 more pennies. How many pennies does he have altogether?" a child was likely to set out 3 Unifix cubes to stand for the 3 pennies, then 6 more cubes to stand for the 6 additional

pennies. Actions performed on the cubes were presumably the child's interpretation and representation of the action or relationships between the sets described in the problem. The second use of cubes occurred when the child initiated a counting sequence beginning with some number word other than "one." Such a counting sequence requires keeping track of the number of counting words in the sequence. For example, if a child begins a forwards sequence at "eight" and counts on to "thirteen," a set of 5 cubes might be set out one by one. Those 5 cubes do not represent a set of 5 objects as given in the original problem situation, but rather counters to keep track of the number of words in the sequence "nine, ten, eleven, twelve, thirteen."

(F) Fingers - Fingers are used to represent sets given in the problem situation or as a tracking device to remember the numbers in a counting sequence.

A number of children did not use any observable physical representation to help solve a problem. They either tried to figure out the problem in their heads or did not understand the problem sufficiently to know how to represent it physically. Such children were coded as:

(N) No physical representation - There was no observable use of cubes or fingers.

Results for addition and subtraction. The actual problems presented to the subjects were listed in Tables 1 and 2 presented earlier. The

summary of results for both the verbal and concrete presentations of addition problems is given in Table 3.

Results for the joining and part-part-whole problems reveal almost identical patterns of behavior. On the other hand, the results from the two comparison problems show that these problems were seen as different. Many subjects simply gave one of the numbers in the problem as the answer and consequently had no need to represent the described relationship between sets.

The results for the subtraction problems are presented in Table 4. In general, most children used cubes to represent the problem set. This tendency was slightly higher, in the concrete problem presentations. A larger number of students used cubes for the two equalizing problems and for the comparison problem than for the other four problems. A possible reason for this difference is that neither of the two sets described in the problems is a subset of the other. It is difficult to represent both sets using fingers or to keep track of operations on both sets without some form of concrete representation.

The part-part-whole problem in the concrete presentation deserves mention. This problem was constructed faithfully according to the structure of the verbal counterpart. Consequently, it was trivial for most children, who simply counted the set of red cubes presented. Since the problem gave no insight into children's problem-solving strategies, it was dropped from analysis.

Correct vs. incorrect, and types of errors. One of the primary objectives of the study was to identify how successful children are

Table 3

Mode of Representation for Addition Problems

Problem	Verbal presentation			Concrete presentation		
	C	F	N	C	F	N
Joining	23	7	13	33	1	9
Part-part-whole	24	6	13	32	4	7
Comparison	17	3	23	19	2	22

Table 4

Mode of Representation for Subtraction Problems

Problem	Verbal Problems			Concrete Problems		
	C	F	N	C	F	N
Separating	25	5	13	31	1	11
Joining 1	27	3	13	31	0	12
Joining 2	27	5	11	28	2	13
Part-part-whole	26	4	13	Dropped from analysis		
Comparison	31	2	10	34	0	9
Equalizing +	32	1	10	31	2	10
Equalizing -	32	3	8	35	0	8

at solving different types of addition and subtraction problems prior to formal instruction in these operations. In other words, one purpose of this study was to determine whether children can independently generate solutions to certain addition and subtraction problems and to identify which types of problems are most difficult for them. This information should provide a basis for deciding which types of problems children readily understand as initial models of addition and subtraction.

Responses were coded correct or incorrect depending upon whether an appropriate strategy was used and whether the strategy resulted in a correct answer. An appropriate strategy is a strategy which would produce a correct answer if properly applied and followed through to its logical conclusion. For many of the problems, more than one strategy could be used to produce a correct answer. The following coding system was employed to record this category:

(V) Validity - A valid or appropriate strategy was used.

(A) Answer - The correct answer was found.

It is possible to record a child as having achieved a correct answer without recording the use of a correct strategy. In those instances, the interviewer was simply unable to determine which strategy a child used.

For both addition and subtraction problems two types of errors were identified.

(CE) The child used a correct strategy but miscounted or perhaps used a wrong number, forgetting one of the original numbers presented in the problem. In either case, a

wrong answer would be generated. If a child happened to miscount twice and have the two errors cancel each other out, arriving at a correct numerical answer, an error was still recorded under this category.

(G) Given number - A child responds that the answer is one of the two numbers given in the original problem.

(O) Wrong operation - A child uses an addition strategy or the given answer strongly indicates that an addition process or basic fact was used.

(E) This category includes use of other incorrect or inappropriate strategies, an unidentifiable strategy with an incorrect answer, an incorrect guess, or failure to generate an answer of any kind.

The results for the addition problems are presented in Table 5.

Overall, subjects were extremely successful in solving both the joining and part-part-whole addition problems. For each problem more than 88% of the subjects used a correct strategy (V), and over 80% found the correct answer (A). The comparison problems turned out to be much more difficult. In the verbal comparison problem, 23 subjects gave one of the given numbers as their response. They did not seem to be able to understand "Jeff had 5 more pieces of gum than Ralph" and interpreted it as "Jeff had 5 pieces of gum." Children could deal with the "more than" relation in the subtraction comparison problem and the two equalizing problems. It may be that for children of this age "more" implies a comparison of two sets, and they cannot understand it

Table 5

Correct and Incorrect Procedures and Error Types for Addition Problems

Problem	Verbal Problems						Concrete Problems					
	Correct		Error				Correct		Error			
	V	A	CE	G	O	E	V	A	CE	G	O	E
Joining	41	34	7	1	1	0	43	40	3	0	0	0
Part-part-whole	38	37	1	1	1	3	41	37	4	0	0	2
Comparison	12	10	2	23	3	5	21	20	1	3	4	15

in terms of incrementing a given set, as in the addition comparison example.

The patterns of responses were almost identical for the joining and part-part-whole problems. In contrast the two comparison problems were not so similar. Ten more subjects correctly solved the concrete problems than solved the verbal problem. Furthermore only 3 subjects gave one of the given numbers as their response to the concrete problems as opposed to 23 for the verbal problem. The subtraction results are presented in Table 6.

On the whole, children were not quite as successful with the subtraction problems as they were with the addition problems. However, over three-fourths of the subjects used the correct strategy, and well over half the responses were correct for every item.

Except for the joining 1 problem, children were about as successful in generating a correct strategy for verbal problems as they were for the corresponding concrete problems. There were slightly more counting errors in verbal problems, but this is to be expected. The verbal problems offered more opportunity to make a counting error since subjects had to construct both initial sets used to generate a solution. In the concrete problem, on the other hand, certain sets were given as part of the problem and subjects were corrected if they made an initial error in counting them.

Contrary to the findings of previous research with older children, very few children used the wrong operation. The most common error was to respond one of the given numbers but there were at most 6 out

Table 6

Correct and Incorrect Procedures and Error Types for Subtraction Problems

Problem	Verbal Problems						Concrete Problems					
	Correct		Errors				Correct		Errors			
	V	A	CE	G	O	E	V	A	CE	G	O	E
Separating	39	31	8	1	0	3	42	40	2	0	0	1
Joining 1	31	22	7	1	5	4	41	38	3	2	0	0
Joining 2	36	31	5	1	1	5	33	31	2	1	0	9
Part-part-whole	33	20	13	6	0	4	Dropped from analysis					
Comparison	35	29	6	3	0	5	39	32	7	3	0	1
Equalizing +	39	30	9	0	0	4	35	28	7	1	0	7
Equalizing -	39	29	10	0	0	4	38	34	4	1	0	4

of 43 instances of this error for any problem.

Strategies

The second major objective of the study was to characterize the processes or strategies that children use to solve different problems and identify the factors that lead to the selection of different strategies. One hypothesis is that children develop single strategies for addition and subtraction and use them in all appropriate problems. For example, a child might use a separating strategy to solve all subtraction problems. A competing hypothesis is that children's strategies match a problem's structure and model the implied actions or relationships in the problem. Different strategies imply different conceptions of addition and subtraction, and identifying the processes that children use to solve different problems should provide some insight into their understanding of addition and subtraction operations.

In the subsections that follow, results are presented for the addition tasks and for several categories of subtraction tasks. Following the two-by-two matrix presented earlier, several different categories of verbal problems were identified. From among these, three separate types of subtraction problems emerge, each having a distinct semantic structure. These three types will be discussed separately.

Addition strategies. The three basic counting models identified by Groen and Parkman (1972) were also found in this study. Several strategies that were not based on counting were also identified.

(CA) Counting all - The counting all strategy can be carried out using cubes or fingers as models, or by counting mentally. If cubes are used, both sets are represented, and then the union of the two sets is recounted beginning with "one". If counting is done mentally or with fingers, the counting sequence begins with "one" and ends with the number representing the total of the two given quantities.

(CF) Counting on from first number - In this strategy, the counting sequence begins either with the first (smaller) given number in the problem or the successor of that number. Counting may be done mantally, or by using cubes or fingers as models.

(CL) Counting on from larger number - This is similar to the previous strategy except that the counting sequence begins with the larger (second) given number or with the successor of that number.

(KF) Known fact - The child gives an answer with the justification that it was the result of knowing some basic addition fact.

(H) Heuristic - Heuristic strategies are employed to generate solutions from a small set of known basic facts. These strategies usually are based on doubles or numbers whose sum is 10. For example, to solve a problem representing 6 + 8 = ? a subject responds that 6 + 6 = 12 and 6 + 8 is just 2 more than 12. In another example involving 4 + 7 = ? a subject responds that 4 + 6 = 10 and

4 + 7 is just 1 more than 10.

(U) Uncodable - A correct answer is provided but the interviewer is unable to determine what strategy a child is employing.

The strategy results for the six addition tasks are presented in Table 7.

The data clearly indicate that the children treated the joining and part-part-whole problems in essentially the same manner. The fact that joining problems have an action component as opposed to the static condition of the part-part-whole problems does not appear to make any difference, at least in the way that children attempt to solve them. When these data are considered together with the correct vs. incorrect data from Table 5, it appears that the two addition problems--joining and part-part-whole--presented no difficulties for the subjects of this study. On the other hand, the comparison problems were extremely difficult.

Although the data in Table 7 do not present the information directly, an analysis was made to determine if a particular strategy was linked to a particular type of representation modality. Results showed that different strategies did tend to be paired with different modes of representation. Almost all students who used cubes used a counting all strategy (CA). For example, to solve the problem that represents 3 + 8 = $\square$, subjects would generally construct a set of 3 cubes, then a set of 8 cubes, and then count the number of cubes in the union of the two sets. They did not even take advantage of the fact that they had already counted both the set of 3 and the set of 8 and did not need to recount

Table 7

Strategies Employed for Addition Problems

Problem	Strategy					
	CA	CF	CL	H	KF	U
Verbal Problems						
Joining	21	4	4	5	1	6
Part-part-whole	22	3	9	2	1	1
Comparison	7	1	1	2	0	1
Concrete Problems						
Joining	26	1	7	5	2	2
Part-part-whole	26	2	9	3	1	0
Comparison	6	4	6	3	0	2

them. In fact, in counting the union of the two sets, many subjects were very careful to count one set first and then the other. If a subject constructed both sets but did not recount them both, the response was coded as the appropriate counting on strategy. There were only 4 such responses in all 3 verbal problems for a total of 14 in the 3 concrete problems.

Counting on from the first number given in the problem (CF) and counting on from the larger number (CL) were the dominant counting strategies for subjects who used fingers (F) or no physical model (N). Only three subjects who used fingers in any of the six problems used a counting all (CA) strategy, and only one who used no model gave a counting all explanation. This is not especially surprising since the counting all process is rather difficult to keep either in one's head or on fingers. Furthermore, it is not unreasonable that the ability to deal with numbers without concrete referents is related to the ability to use the more abstract counting on strategies.

In addition to the three counting strategies, a heuristic strategy (H) was employed by a few children to generate solutions from a small set of known basic facts. Heuristic strategies always involved no physical modeling.

Several subjects knew the addition fact required to solve the given problem (KF) and there were a few responses that were uncodable (U). A response was uncodable only if a subject got a correct response but the experimenter was unable to determine the strategy. The uncodable category assumes that the subject used an appropriate strategy but was unable to explain the process.

Subtraction strategies. Four basic subtraction responses were identified. They take on a different form depending upon the representation model chosen. For concrete representations they are:

(S) Separating - The child models the larger given set and then takes away or separates, one at a time, a number of cubes equal to the given number in the problem. Counting the set of remaining cubes yields the answer.

(ST) Separating to - After the larger set is modeled, the child removes cubes one at a time until the remainder is equal to the second given number of the problem. Counting the number of cubes removed gives the answer.

(AO) Adding on - The child sets out a number of cubes equal to the smaller given number (an addend). The child then adds cubes to that set one at a time until the new collection is equal to the larger given number. Counting the number of cubes added on gives the answer.

(M) Matching - The child puts out two sets of cubes, each set standing for one of the given numbers. The sets are then matched one-to-one. Counting the unmatched cubes gives the answer.

Three more abstract counting strategies were also observed. These are the analogues to the first three concrete strategies listed above.

(CB) Counting back - A child initiates a backwards counting sequence beginning with the given larger number. The back-

wards counting sequence contains as many counting number words as the given smaller number. The last number uttered in the counting sequence is the answer. This is the counting analogue to the separating (S) strategy.

(CT) Counting back to - A child initiates a backwards counting sequence beginning with the larger given number. The sequence ends with the smaller number. By keeping track of the number of counting words uttered in this sequence, either mentally or by using fingers or cubes, the child determines the answer to be the number of counting words used in the sequence. This is the counting analogue to the separating to (ST) strategy.

(CU) Counting up from smaller - A child initiates a forward counting sequence beginning with the smaller given number. The sequence ends with the larger given number. Again, by keeping track of the number of counting words uttered in the sequence, the child determines the answer. This is the counting analogue to the adding on (AO) strategy.

The known fact (KF), heuristic (H), and uncodable (U) categories follow the same rules as the corresponding addition categories.

Certain of the strategies naturally model the action described in specific problems. The separating problem is most clearly modeled by the separating (S) strategy or the related counting (CB) strategy. On the other hand, the implied joining action of the joining (missing

addend) problems is most closely modeled by the adding on (AO) and counting up (CU) strategies. Comparison problems, on the other hand, deal with relationships between sets rather than action. In this case the matching strategy (M) appears to provide the best model.

For the part-part-whole and equalizing problems the situation is more ambiguous. In the part-part-whole problems there is no implied action so neither the separating or adding on strategies seem more appropriate. But since one of the given quantities is a subset of the other, there are no two distinct sets that can be put into one-to-one correspondence.

For the equalizing problems the situation is reversed. Since the equalizing problems involve both a comparison and some implied action, two different strategies might be seen as appropriate. The addition equalizing problems involve a comparison of two quantities and a decision of how much should be joined to the smaller quantity to make them equivalent. Thus, both the matching (M) or the adding on (AO) and counting up from smaller (CU) strategies might be appropriate. For the subtracting equalizing problems the implied action involves removing elements from the larger set until the two sets are equivalent. This action seems to be best modeled by the separating to (ST) and counting back to (CT) strategies while the matching strategy (M) is again appropriate for the comparison aspect of the problem.

Verbal problems. For verbal problems, problem structure does appear to be the major determinant of solution strategy (Table 8). For the separating problem almost three times as many subjects use a separating (S) or counting back (CB) strategy as used all the other strategies combined.

Table 8

Strategies for Verbal Subtraction Problems

Problem	Strategy									
	S	ST	AO	M	CB	CT	CU	H	KF	U
Separating	19	0	0	4	9	0	0	3	2	2
Joining 1	4	0	15	2	1	0	2	3	0	2
Joining 2	4	1	14	5	0	0	8	3	0	1
Part-part-whole	9	0	10	4	3	0	3	0	2	2
Comparison	8	1	2	17	0	0	3	3	1	0
Equalizing +	9	2	5	15	0	1	1	5	0	1
Equalizing -	14	4	0	13	1	3	2	2	0	0

For the two joining problems, the pattern of responses was almost identical. For each problem the adding on (AO) or counting up (CU) strategies were used almost twice as often as all the other strategies combined. With the comparison problem, matching (M) was the dominant strategy.

The ambiguity of the part-part-whole problem is reflected in the children's strategies which were about evenly divided between separating (S) and adding on (AO). Support for our analysis of the equalizing problem is less strong, but it is generally consistent with the proposed model. Matching was a dominant strategy for both equalizing problems, but in both cases separating was used more frequently than the hypothesized separating to or adding on strategy. However, a comparison of the two equalizing problems reveals that adding on (AO and CU) was used more frequently than separating to (ST and CT) for the addition problem (six cases and three cases respectively) while the reverse was true for the subtraction problem (two and seven cases respectively).

Concrete problems. For the concrete problems, the problem structure analysis does not predict performance nearly as well (Table 9). For four of the six problems, separating (S) was the principle strategy and for another separating (S) and separating to (ST) were employed with almost equal frequency. The only problem for which separating was not the dominant strategy was the second joining problem. One explanation for this pattern of responses is that strategies were determined by the characteristics of the set of cubes subjects had available when they began to solve the problem. This is clearly illus-

Table 9

Strategies for Concrete Subtraction Problems

Problem	Strategy									
	S	ST	AO	M	CB	CT	CU	H	KF	U
Separating	30	1	0	0	7	0	0	1	1	2
Joining 1	23	6	0	0	3	0	0	5	1	3
Joining 2	7*	0	16	0	0	0	4	2	1	3
Comparison	23	4	0	5	0	0	2	4	1	0
Equalizing +	18	3	4	2	2	0	2	2	1	1
Equalizing -	15	15	0	2	1	1	0	1	2	1

*Includes 5 responses that subjects started to add on and then switched to separating strategy.

trated by the contrast between the strategies used to solve the two joining problems. Although they both describe essentially the same action, they provide subjects with very different starting points. For example, consider the problem 11 - 3 = ?. In the first case the experimenter shows a subject a set of 3 cubes, adds some more cubes, and asks the subject to determine how many cubes were added. In the second example, a subject is given a set of 3 cubes and asked how many more are needed to have 11 cubes altogether. The key difference between the two problems is that in the first case the subject has 11 cubes to start with and can find the answer by simply removing 3 cubes. In the second case they must first construct the set of 11 cubes which is easiest to do by simply adding on to the set of 3. For every problem by the one joining problem, subjects had the larger set available and consequently relied primarily on a separating strategy.

Although problem structure was not the primary determinant of subjects' solution strategy it did appear to have some effect. The only use of the matching strategy (M) occurred with the comparison problem and one equalizing problem, which is consistent with the analysis of problem structure. Comparing the two equalizing problems reveals that six subjects used an adding on strategy (AO and CU) for the addition problem while none used it for the subtraction problem. On the other hand, 16 subjects used the separating to strategy (ST and CT) with the subtraction problem while only 3 used it for the addition problem. Both of these results are consistent with the analysis of the two equalizing problems.

One of the more interesting differences between the set of concrete problems and the set of verbal problems involves the use of the matching strategy. The matching strategy was used for every verbal problem at least twice and was a primary strategy for the comparison problem and the two equalizing problems. In the concrete set, however, it was only used on three problems for a total of nine times. It is not surprising that the matching strategy was not used for the joining or separating problems, where it would have been necessary to construct the second set. But for the comparison and equalizing problems both sets were already constructed. It is not clear why children would go to the trouble of constructing two sets to use a matching strategy in the verbal case and not use a matching strategy in the concrete case, when the sets are already constructed. The matching strategy is actually more efficient for concrete problems than for verbal problems.

The most prevalent strategy overall was clearly the separating (S) strategy. Although the use of this strategy was not as overwhelming for verbal problems as concrete problems, it was still the most commonly used strategy. It was the only strategy that was frequently used in contexts that were inconsistent with the analysis of problem structure. The choice of numbers in the subtraction problems (11 - 3 = ? rather than 11 - 8 = ?) may have created some bias in favor of separating and counting back strategies, which may in part account for the popularity of the separating strategy. But no subjects indicated that number size influenced their choice of strategy. On the whole there is no basis for concluding that the choice of numbers had any influence on children's strategies. However, this is one limitation of this

study, and additional research would be required to demonstrate conclusively that relative number size had no effect.

On the whole children were not quite as successful with the subtraction problems as they were with the addition problems. But over three-fourths of the subjects used the correct strategy, and well over half the responses were correct for every item. Furthermore, no one problem stood out as significantly more difficult than the others. Contrary to the findings of previous research with older children, very few children used the wrong operation. The most common error was to respond one of the given numbers but this accounted for at most six responses for any given problem.

Patterns of Children's Responses

This section focuses on the responses of individual children or groups of children over sets of related problems. Our objective is to attempt to identify groups of children who apply similar strategies over several problems and to characterize their pattern of responses.

The different combinations of responses for the joining and part-part-whole addition problems are summarized in Table 10. Twenty-six subjects used the same strategy for both verbal problems and 25 used the same strategy for both concrete problems. Thus, although the two problems have very similar patterns of responses (Tables 3, 5, and 7), just over half the subjects used the same strategy for both problems.

It is a bit more difficult to identify general strategies for solving the subtraction problems because there are more problems and more possible strategies for each. Most of the subjects' general strategies were defined in terms of the strategies used on individual problems. For example, a subject would be classified as using a general separating strategy if the subject used this strategy on most problems, regardless of their structure. A second major type of general strategy appeared to be based on problem structure. A subject was classified as using a problem structure strategy if the strategies predicted by our logical analysis of the problem structure were generally used in that subject's solutions. Our decision rule was to

Table 10

Responses for Joining and Part-part-whole Addition Problems

Response combinations	Verbal	Concrete
Both counting all	17	20
Both counting on	6	3
Both heuristic	3	2
Counting all-counting on	4	8
Heuristic-counting on	3	4
Heuristic-counting all	0	2
Other	10	4

classify a subject as using a particular strategy if the subject used the strategy for five of the seven verbal problems or used it for four problems and used a number fact, heuristic, or uncodable strategy for the others. For the concrete problems the decision rule was four out of six, with the part-part-whole problem again excluded from the analysis.

Over half of the subjects could be identified as using a particular general strategy (Table 11). The results are consistent with results for individual problems. The most frequently used general strategy for verbal problems was problem structure and for concrete problems it was separating.

In the analysis of individual subtraction problems several problems showed similar solution patterns (Tables 4, 6, and 8). The two joining missing addend problems (problems 5 and 6) had almost identical patterns of response. However, an analysis of individual subjects' responses reveals that although the overall pattern of responses was similar, subjects were not consistent in responding to the two problems. Only 13 subjects used the same strategy for problems 5 and 6. The two equalizing problems and the comparison problem had similar patterns of response. For these 3 problems, 13 subjects used the same strategy on all 3 problems and an additional 18 used the same strategy on 2 out of 3 problems.

The use of the more sophisticated strategies is also of interest. Almost a third of the subjects used a heuristic strategy at least

Table 11

Classification of Subjects' General Subtraction Strategies

General strategy	Verbal	Concrete
Single strategy		
Separating	3	13
Matching	4	0
Heuristic	2	3
Structure of problem	14	6
Consistent error	3	3
Unclassifiable	17[a]	18

[a]Includes eight subjects who used only two strategies, four of whom used AO and S.

once, and almost three-fourths used at least one of the more advanced strategies (heuristic, counting up, or counting back).

An analysis was done of the number of errors associated with various strategies. The errors were fairly evenly divided over all strategies. The number of errors for each strategy was roughly proportional to the number of times the strategy was selected. Thus, there is no evidence that any strategy is more reliable than any other.

To check for any effect of the order in which problems were administered, the strategies of the group of subjects who received the verbal problems first were compared to the strategies of the group receiving concrete problems first. For the most part the patterns of responses were virtually identical. But for the two verbal joining missing-addend problems and the verbal part-part-whole subtraction there was a consistent difference. For each of these three problems, almost twice as many of the concrete-first subjects as verbal-first used cubes, used an add-on strategy and calculated the answer. It is to be expected that some problems would show marked differences simply by chance. But these three problems also tended to generate similar patterns of solution (see Tables 4, 6, and 8), being the only verbal problems for which add-on was the primary strategy. It is difficult to explain why differences should only occur for this cluster of three problems.

Conclusions

A striking result of this study is the high level of success of first-grade children in solving verbal problems. Only four subjects used an incorrect strategy for more than half of the verbal problems, and over two-thirds used a correct strategy for 8 of the 10 problems. Children were not only successful in interpreting action or relationships implied in problems. They were also able to use different models of addition and subtraction when convenient and demonstrated some understanding of the inverse relationship between addition and subtraction.

The first-grade children in this study gave very little evidence of the types of systematic errors reported in previous studies. Very few used the wrong operation in their solutions. Since this error has been observed primarily with older children who have already experienced formal instruction in addition and subtraction, it may be a result of learning symbolic representations. In typical classroom procedure, addition and subtraction are introduced in terms of joining or separating sets using either pictures or concrete objects. Then children are drilled on abstract problems with number sentences. When they finally get to verbal problems, their response is, "Is this a plus or a takeaway?" In this format the operations are initially learned outside the context of verbal problems. When verbal problems

are introduced later, children are simply told that addition and subtraction can be used to solve these problems, but they have no basis for using their natural intuition to relate the problem structure to the operations they have learned. In other words, their natural analytic problem solving skills are bypassed, and they too often resort to superficial problem characteristics to identify the correct operation. This may result not only in a limited understanding of addition and subtraction but also in a decline in general problem-solving ability.

The results of this study suggest a somewhat different picture of children's processes for solving addition and subtraction problems than has been proposed in other analyses. Greeno (Note 2) hypothesizes that children associate solution strategies directly with the semantic content of problems rather than constructing sets of simultaneous equations based on syntactic information within the problem. Greeno's analysis is consistent with the results of this study. However, Greeno also hypothesizes that some problems are associated directly with an operation while others are first transformed to different structures. Specifically, joining missing addend problems and certain comparison problems are first transformed to part-part-whole problems.

The results of this study suggest a different hypothesis. The tremendous variability between children as well as the variety of processes observed suggest that before formal instruction, young

children do not transform problems into a single type and apply a single strategy. The results indicate that children have a rich repertoire of strategies available and that they make use of many of these to solve various problems. It is still not clear what triggers the use of a particular strategy, but it seems plausible that children solve each problem type directly, rather than collapsing them and applying a single strategy consistently.

The picture painted by this description is quite different from that proposed by Greeno. In Greeno's description, the limiting factor is the number of different solution strategies children have available. Since empirical data show that children can solve a variety of problem types it was assumed that they must transform them in order to successfully apply the few strategies which they have in their repertoires. The results presented here suggest that even prior to instruction most children possess numerous different strategies and select from among them a method appropriate to solve each problem type directly. No transformations are needed. In fact, it may be the transformation process itself which is the limiting factor as children begin instruction.

Arithmetic instruction frequently illustrates a particular operation like subtraction with several problem types (e.g., separating, part-part-whole, comparison). Although our findings show many children can solve each type of problem using an appropriate strategy (e.g., separating, add on, matching), they may have trouble transforming

these problems and understanding that a single strategy could be appropriate for all of them. This conjecture is supported by the small number of children in this study who used a single strategy consistently across problem types, and by the well documented difficulties which children experience with missing addend problems in most curriculum programs.

The results of this study also deviate to some degree from the results of earlier latency studies of children's solution of number sentences (Groen & Parkman, 1972; Groen & Resnick, 1977; Woods, Resnick, & Groen, 1975). Specifically our study found less frequent use of counting on strategies for addition problems than in earlier studies (Groen & Parkman, 1972; Groen & Resnick, 1977), and although there was no direct test of the effect of number size, other factors than this one seemed to have a greater influence in determining children's choice between adding on or counting back strategies. This study also identified two strategies, matching and heuristic, that were not even considered in the earlier studies.

To some extent these discrepancies may result from differences in the age of subjects and in characteristics of the problems. Certainly it is necessary to be very careful in making comparisons between the solution of verbal problems and the solution of number sentences. Two factors that may contribute to the differences in performance are the different number domains and the availability of cubes. The larger numbers in this study perhaps make counting on or choice strategies less likely. The availability of cubes clearly seems to promote

a counting all rather than a counting on strategy. This is illustrated by the fact that those who used the cubes almost always used a counting all strategy while those using fingers or no action generally used a counting on strategy. To some extent this may result from the fact that the more capable children, those most able to use more sophisticated counting on strategies, tended to be the ones who did not use cubes. But the cubes did appear to encourage children to model the complete problem.

One of the most fundamental differences between this study and the earlier studies is in the experimental paradigm: clinical interview as opposed to matching response latencies to predicted regression equations. The response latency paradigm assumes that children consistently apply a well defined strategy to their solution problems. The results of this study indicate that this assumption is at least suspect, and the results of response latency studies should be subjected to further validation. It also appears that one should be very careful in generalizing the results of any research of this type beyond the domain of problems included in the specific study.

The results of this study tend to support the hypothesis that verbal problems may be the most appropriate context in which to introduce addition and subtraction operations. Clearly, verbal problems are a viable alternative to traditional approaches since children are able to interpret and solve them prior to formal instruction. Verbal problems also provide different interpretations of addition and subtraction, interpretations that are important for children to under-

stand. Perhaps by basing our introduction of operations on verbal problems and integrating verbal problems throughout the mathematics curriculum rather than using them only as an application of previously taught algorithms, we can allow children to develop their natural ability to analyze problem structure and to develop a broader concept of basic operations.

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